

THE HYPEROSCULATING POINTS OF THE GENERALIZED FERMAT CURVES

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ABSTRACT. Let (S, H) be a generalized Fermat curve of the type (k, n) and let $F \subset S$ be the set of the fixed point by H . In this article we will show that F is exactly the set of hyperosculating points of the standard embedding of S , and we will give an lower bound (non trivial) for the Weierstrass weight of these points.

1. INTRODUCTION

It is known the the geometry of compact Riemann Surfaces can be described via projective algebraic curves, Fuchsian group, Schottky groups, Abelian varieties, etc. However, given one of these descriptions, explicitly obtaining the others is a difficult problem, in fact in general it is a problem that has not been solved. The majority of examples of Riemann Surfaces where we explicitly know the uniformizing Fuchsian group, and the equations of an algebraic curve which represents them, are rigid examples, in other words they are not families. The generalized Fermat Curves of the type (n, k) form a family of Riemann surfaces of complex dimensions $n - 2$ in which we explicitly know, for each member of the family, a representation as an algebraic curve, and the uniformizing Fuchsian group. A generalized Fermat curve of the type (k, n) is an couple (S, H) where S is a compact Riemann surface and H is a subgroup of the conformal automorphisms group of S (in which we denote for $\text{Aut}(S)$), such that $H \cong \mathbb{Z}_k^n$ and the natural morphism $S \rightarrow S/H$ has exactly $n + 1$ branch points, and each ramification point has a ramification index equal to k . The group H is called the generalized Fermat group of the type (k, n) .

Let R be a marked Riemann surface (Orbifold) of type $(0, n + 1, k)$, i.e. R is the Riemann sphere equipped with $n + 1$ distinct points marked with the integer k . The orbifold R is uniformized by a Fuchsian group $\Gamma \cong \langle x_1, \dots, x_{n+1} \mid x_1^k = \dots x_{n+1}^k = x_1 \dots x_{n+1} = 1 \rangle$. Observe that $\Gamma/[\Gamma : \Gamma] \cong \mathbb{Z}_k^n$, and that the group $[\Gamma : \Gamma]$ is torsion-free. Then this group uniformizes a generalized Fermat curve of the type (k, n) , what's more, all generalized Fermat curves are obtained in this manner. In fact, this construction is unique, which is to say considering an orbifold of the type $(0, n + 1, k)$, the respective generalized Fermat curve is the maximal abelian covering space of the orbifold.

On the other hand, let us consider the projective smooth curve given by the following system of equations:

$$S := \left\{ \begin{array}{ccc} x_0^k + x_1^k + x_2^k & = & 0 \\ \lambda_1 x_0^k + x_1^k + x_3^k & = & 0 \\ \vdots & \vdots & \vdots \\ \lambda_{n-2} x_0^k + x_1^k + x_n^k & = & 0 \end{array} \right\} \subset \mathbb{P}^n,$$

where $\lambda_i \neq 1, 0$ for all $i \in \{1, 2 \dots n - 2\}$ and $\lambda_i \neq \lambda_j$ for all $i \neq j$. Let H be the subgroup of $\text{Aut}(S)$ generated by the following transformations:

$$\varphi_j([x_0 : \dots : x_j : \dots : x_n]) := [x_0 : \dots : w_k x_j : \dots : x_n], \text{ where } w_k := e^{\frac{2\pi i}{k}}.$$

Note that $H \cong \mathbb{Z}_k^n$ and that each fiber of the morphism $S \rightarrow \mathbb{P}^1 : [x_0, \dots, x_n] \mapsto [-x_1^k, x_0^k]$ is exactly an orbit for the action of the group H , in particular we can prove that S is a generalized Fermat curve of the type (k, n) . It stands out that we can explicitly obtain the branch points as a function of the vector $(\lambda_1, \dots, \lambda_{n-2})$. In this case, the correspondence between Fuchsian groups and algebraic curves is obtained in this manner.

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The embedding $S \subset \mathbb{P}^n$ is called the standard embedding of the generalized Fermat curve. Note that the generalized Fermat curves of the type $(k, 2)$, $k \geq 4$ are exactly the classic Fermat curves. For more details see [GDHL09].

In the article previously cited, the following question is raised: Given a generalized Fermat curve of the type (k, n) , is the generalized Fermat group of the type (k, n) unique? The affirmative answer to this question implies that the moduli space of the orbifolds of the type $(0, n, k)$ parametrizes (injectively) the generalized Fermat curves of the type (k, n) , and also facilitates the calculation of the extra automorphisms. An affirmative answer for the case $(k, 3)$ was obtained in [FGDHL13]. Recently in [KP14] an affirmative answer for the general case was announced.

In this work we will begin the study of the hyperosculating points and the Weierstrass points of the generalized Fermat curves. These points are important in the geometry of Riemann surfaces, and in general, considering a surface, in determining all the Weierstrass points, and their respective weights, which remains a difficult problem.

In the case of the classic Fermat curves, in 1950 Hasse ([Has50]) calculated the Weierstrass weight of the hyperosculating points. In 1982 Rohrich ([Roh82]) calculated for $k \geq 5$ explicitly $3k^2$ new Weierstrass points of this curve. In 1999 Watanabe ([Wat99]) showed that in the case $k = 6$, additional Weierstrass points exist. The Weierstrass weight of points fixed by involutions in the case $k = 9, 10$ was calculated by Towss in [Tow00].

In this article we prove that the hyperosculating points of a generalized Fermat curve are exactly the fixed points of the generalized Fermat group, and we calculate a lower bound for the Weierstrass weight (it is known that the fixed points of the generalized Fermat group are Weierstrass points).

2. HYPEROSCULATING POINTS.

Let (S, H) be a generalized Fermat curve of the type (k, n) . Remember that from the description presented in [GDHL09] we may suppose that S is the smooth curve given by the following system of equations:

$$S := \left\{ \begin{array}{ccc} x_0^k + x_1^k + x_2^k & = & 0 \\ \lambda_1 x_0^k + x_1^k + x_3^k & = & 0 \\ \vdots & \vdots & \vdots \\ \lambda_{n-2} x_0^k + x_1^k + x_n^k & = & 0 \end{array} \right\} \subset \mathbb{P}^n,$$

where $\lambda_i \neq 1, 0$ for all $i \in \{1, 2, \dots, n-2\}$ and $\lambda_i \neq \lambda_j$ for all $i \neq j$, and that H is the subgroup of $\text{Aut}(S)$ generated by the following transformations:

$$\varphi_j([x_0 : \dots : x_j : \dots : x_n]) := [x_0 : \dots : w_k x_j : \dots : x_n], \text{ where } w_k := e^{\frac{2\pi i}{k}}.$$

Observe that the fixed points of φ_j in S are given by the intersection $\text{Fix}(\varphi_j) := F_j \cap S$, where F_j is the hyperplane $\{x_j := 0\} \subset \mathbb{P}^n$. Let $F := \cup_{j=0}^n \text{Fix}(\varphi_j)$. Observe that $F = \text{Fix } H$.

In the Section 2.2 we will prove that the set of hyperosculating points of the generalized Fermat curve S is exactly the set F . Let $p \in F$. It is known that the point p is a Weierstrass point of S (see article cited above). In Section 2.3 we will obtain a lower bound for the Weierstrass weight $w(p)$.

Before studying the hyperosculating points of the generalized Fermat curves, we will briefly review the general theory of the Plucker formulas in the case of smooth curves.

2.1. Preliminaries. The purpose of this section is not to review extensively the theory, but to present a self contained overview of the parts of the theory relevant to us. All the results presented in this section can be found in [GH94].

Let C be a projective smooth curve of the projective space \mathbb{P}^n . Let us consider an l -plane $P \subset \mathbb{P}^n$, $1 \leq l \leq n-1$, and let us define the multiplicity of P in p :

$$\text{mult}_p(P \cap C) := \text{Order of contact of } P \text{ and } C \text{ in } p$$

It is known that there exists a unique l -plane, denoted by $P(l, p)$, such that $\text{mult}_p(P(l, p) \cap C) \geq l+1$, and that there exists at most a finite number of points $p \in C$ such that $\text{mult}_p(P(l, p) \cap C) > l+1$.

Definition 1. A point $p \in C$ is called a *hyperosculating point* if

$$\text{mult}_p(P(n-1, p) \cap C) > n.$$

Remark 2. Let $\varphi \in \text{Aut}(\mathbb{P}^n) \cong \text{PGL}(n+1, \mathbb{C})$. Observe that

$$\text{mult}_{\varphi(p)}(\varphi(P) \cap \varphi(C)) = \text{mult}_p(P \cap C).$$

In particular, p is a hyperosculating point of C if and only if $\varphi(p)$ is a hyperosculating point of $\varphi(C)$.

Remark 3. Let S be a non hyperelliptic Riemann surface of genus g , and $f : S \rightarrow \mathbb{P}^{g-1}$ the canonical embedding. Then the hyperosculating points of $f(S)$ are the Weierstrass points of S . More details regarding the above: Let D be a divisor in S and $h(D) := \dim_{\mathbb{C}} \mathcal{L}(D)$, where

$$\mathcal{L}(D) := \{f \text{ meromorphic function on } S \mid (f) + D \geq 0\}$$

Let $p \in S$. Then $h(jp) = h((j-1)p) + d(jp)$, where, $1 \leq j$ and $d(jp) \in \{0, 1\}$. Using the Riemann-Roch formula, it is obtained that $h(K - jp) = h(K - (j-1)p) + d(jp) - 1$, where K is the canonical divisor. Then p is a Weierstrass point of S if and only if there exists $1 \leq j_0 \leq g$ such that $d(j_0p) = 1$. In particular $h(K - j_0p) = h(K - (j_0-1)p)$. Consider the following succession of integers:

$$0 \leq h(K - gp) \leq h(K - (g-1)p) \leq \dots \leq h(K - p) \leq h(K) = g$$

Then p is a Weierstrass point if and only if $h(K - gp) > 0$ which is equivalent to $f(p)$ being a hyperosculating point of $f(S)$.

Given that the l -planes of \mathbb{P}^n are in bijective correspondence with the vectorial subspaces of dimension $l+1$ of \mathbb{C}^{n+1} , we can define the following functions:

$$f_l : C \rightarrow \mathbb{G}(l+1, n+1); p \mapsto P(l, p),$$

where $\mathbb{G}(l+1, n+1)$ is the Grassmannian.

Let $f_0 : C \rightarrow \mathbb{P}^n$ be the natural embedding defined by the inclusion $C \subset \mathbb{P}^n$ and let us consider a local chart $z : U \subset C \rightarrow W \subset \mathbb{C}$, $z(p) = 0$, around the point $p \in C$. Then there exists a neighborhood $W' \subset W$ of 0, and a holomorphic vectorial function

$$v : W' \rightarrow \mathbb{C}^{n+1} \setminus \{0\} : z \mapsto v(z) := (v_0(z), v_1(z), \dots, v_n(z))$$

such that

$$f_0(z) = [v_0(z) : v_1(z) : \dots : v_n(z)] \text{ for all } z \in W'.$$

Let us consider the holomorphic vectorial function

$$w : W' \rightarrow \wedge^{l+1} \mathbb{C}^{n+1} : z \mapsto w(z) := v(z) \wedge v'(z) \wedge \dots \wedge v^{(l)}(z).$$

There exists an integer $k \geq 0$ such that $\frac{w(z)}{z^k}$ is a holomorphic vectorial function which does not vanish in a neighborhood W'' of $z = 0$. By abuse of notation, we can say that $[w(z)] \in \mathbb{P}(\wedge^{l+1} \mathbb{C}^{n+1})$ for all $z \in W''$. Then, using the Plucker coordinates, it is possible to see $\mathbb{G}(l+1, n+1)$ as a subvariety of $\mathbb{P}(\wedge^{l+1} \mathbb{C}^{n+1})$ and that

$$f_l(z) = [v(z) \wedge v'(z) \wedge \dots \wedge v^{(l)}(z)] \text{ for all } z \in W''.$$

In particular, the applications f_l are holomorphic.

The curves $C_l := f_l(C)$, $0 \leq l \leq n-1$ are called the *associated curves* of C . Let us define the following integers:

- $b_l(p)$ is the ramification index of $f_l : C \rightarrow C_l$ in the point $p \in C$.
- $b_l = \sum_{p \in C} b_l(p)$ is the total ramification index of $f_l : C \rightarrow C_l$.
- Let d_l be the number of osculating l -planes of C which intercept a generic $(n-l-1)$ -plane of \mathbb{P}^n . Observe that d_0 is simply the degree of the curve

The following proposition establishes a relationship between the hyperosculating points of C and the ramification indexes of the applications $f_l : C \rightarrow C_l$.

Proposition 4. The point $p \in C \subset \mathbb{P}^n$ is a hyperosculating point if and only if $\sum_{l=1}^{n-1} b_l(p) \geq 1$.

In the case of the generalized Fermat curves we explicitly calculate the ramification indexes, which allows us to determine the hyperosculating points. The following theorem will be useful for this purpose.

Theorem 5 (Plücker Formulas). *Let $C \subset \mathbb{P}^n$ be curve of the genus g . Therefore the following formulas are satisfied:*

$$d_{l+1} - 2d_l + d_{l-1} = 2g - 2 - b_l, \text{ for all } 1 \leq l \leq n-1,$$

where $d_{-1} = d_n = 0$.

Before finishing this section we will show a method for calculating the integers $b_l(p)$.

Returning to the notations defined above, let z be a local chart around the point $p \in C$ and

$$f_0(z) = [v_0(z) : v_1(z) : \cdots : v_n(z)],$$

the application f_0 seen in local charts. Making linear changes of coordinates, it is possible to verify that there exists $\varphi \in \text{Aut}(\mathbb{P}^n) \cong \text{PGL}(n+1, \mathbb{C})$ such that

$$(\star) \quad \varphi(f_0(z)) = [1; z^{1+\alpha_1} + \cdots : z^{2+\alpha_1+\alpha_2} + \cdots : \cdots : z^{n+\alpha_1+\cdots+\alpha_n} + \cdots].$$

By abuse of notation, we will identify $\varphi(f_0(z))$ with $f_0(z)$. The obtained in (\star) is called the *normal form* of f_0 in p . It is possible to verify that the integers α_j , $1 \leq j \leq n$ only depend on f_0 and the point $p \in C$ and neither on the chosen local chart z , nor the vectorial function $v(z)$ nor the automorphism φ .

The following proposition provides a method to calculate the ramification indexes.

Proposition 6. *Retaining the previous notations, we obtain*

$$b_l(p) := \alpha_{l+1}, \quad 0 \leq l \leq n-1.$$

In particular, as C is a smooth curve, it is obtained that $\alpha_1 = 0$.

Remark 7. *Let S be a Riemann surface of genus g and $f : S \rightarrow \mathbb{P}^{g-1}$ the canonical embedding. Then the integers*

$$a_i = i + \sum_{j=1}^{i-1} \alpha_j, \quad 1 \leq i \leq g,$$

*are the gaps values of p . In other words: the a_i are the only g integers where **there does not exist** a meromorphic function of S with a pole of order a_i in the point p .*

2.2. The hyperosculating points of the generalized Fermat curves. Consider the notations fixed at the beginning of the Section 2.

Remember that (S, H) is a generalized Fermat curve of the type (k, n) and that $F := \text{Fix}H$. In the first half of this section we demonstrate that the set F is the set of hyperosculating points of S . In the second half we will obtain a lower bound for the weight of point p .

Let p be a point in F . Using linear substitutions in the system of equations which define the curve S , followed by an automorphism of \mathbb{P}^n , we can suppose that $p \in \text{Fix}(\varphi_1)$, and that S is given by the following system of equations (these transformations do not affect the condition of being or not being a point of hyperosculating, see observation 2):

$$\left\{ \begin{array}{lcl} x_0^k + x_1^k + x_2^k & = & 0 \\ \hat{\lambda}_1 x_0^k + x_1^k + x_3^k & = & 0 \\ & \vdots & \vdots \\ \hat{\lambda}_{n-2} x_0^k + x_1^k + x_n^k & = & 0 \end{array} \right\},$$

where $\hat{\lambda}_i \neq 1, 0$ for all $i \in \{1, 2 \cdots n-2\}$ y $\hat{\lambda}_i \neq \hat{\lambda}_j$ for all $i \neq j$.

In order to simplify the notations, we say that $\hat{\lambda}_0 = 1$. Then the point p in homogeneous coordinates is

$$p := [1 : 0 : \rho_1 : \rho_2 : \cdots : \rho_{n-1}],$$

where $\rho_i^k = -\hat{\lambda}_{i-1}$, $0 \leq i \leq n-1$.

Let $f_0 : S \rightarrow \mathbb{P}^n$ be the natural embedding defined by the inclusion $S \subset \mathbb{P}^n$, and let us consider the following Taylor series centered in $t = 0$:

$$\sqrt[k]{1+t} = \sum_{i=0}^{\infty} \binom{k^{-1}}{i} t^i, |t| < 1,$$

$$\text{where } \binom{k^{-1}}{i} := \frac{\Gamma(k^{-1}+1)}{\Gamma(i+1)\Gamma(i-k^{-1})}.$$

Using this expansion, we can describe f_0 explicitly in a neighborhood of p . Let z be a local chart around p , f_0 can be locally expressed by the following form:

$$f_0(z) = [1 : z : \sum_{i=0}^{\infty} c_{(i,1)} z^{ik} : \sum_{i=0}^{\infty} c_{(i,2)} z^{ik} : \cdots : \sum_{i=0}^{\infty} c_{(i,n-1)} z^{ik}],$$

$$\text{where } c_{(i,j)} := \frac{\rho_j}{\hat{\lambda}_{j-1}^i} \binom{k^{-1}}{i}, 1 \leq j \leq n-1, i \geq 0.$$

The following lemma helps us to find the normal form of f_0 around $z = 0$.

Lemma 8. *Let us conserve the previously defined notations. Then there exists a sequence of $n-1$ integers, $1 = l_0 < 2 = l_1 < l_2 < \cdots < l_j < \cdots < l_{n-2}$, such that the normal form of f_0 around $z = 0$ is the following:*

$$f_0(z) = [1 : z : g_0(z^k) : g_1(z^k) : \cdots : g_i(z^k) : \cdots : g_{n-2}(z^k)]$$

where the g_i are holomorphic functions such that $g_i(z) = z^{l_i} + \cdots$.

Proof. We demonstrate, by induction on j , that for each integer $1 \leq j \leq n-2$, there exists a sequence of $n-2$ integers

$$1 = l_0 < 2 = l_1 < l_2 < \cdots < l_j \leq \cdots \leq \cdots \leq l_{n-2},$$

for which there exists a change of coordinates of \mathbb{P}^n (which is to say, an automorphism of \mathbb{P}^n) such that

$$f_0(z) = [1 : z : \sum_{i=1}^{\infty} d_{(i,1)} z^{ik} : \sum_{i=2}^{\infty} d_{(i,2)} z^{ik} : \sum_{i=l_2}^{\infty} d_{(i,3)} z^{ik} : \cdots : \sum_{i=l_{n-2}}^{\infty} d_{(i,n-1)} z^{ik}],$$

where $d_{(l_{m-1},m)} = 1$ for all $1 \leq m \leq n-2$. Which shows the lemma.

Remembering that for all $\varphi \in \text{Aut}(\mathbb{P}^n)$, $\varphi(S)$ is not contained in a hyperplane coordinate of \mathbb{P}^n . This ensures that the formal series appearing in the coordinates of $\varphi(f_0)$ are not identically zero.

Making a change of coordinates in \mathbb{P}^n , we can suppose that

$$f_0(z) = [1 : z : \sum_{i=1}^{\infty} c_{(i,1)} z^{ik} : \sum_{i=1}^{\infty} c_{(i,2)} z^{ik} : \cdots : \sum_{i=1}^{\infty} c_{(i,n-1)} z^{ik}].$$

Set $a_{(1,m)} := \frac{c_{(1,m)}}{c_{(1,1)}} = \frac{\hat{\lambda}_0 \rho_m}{\hat{\lambda}_{m-1} \rho_1}$. Since $\hat{\lambda}_{m-1} \neq 1$, $2 \leq m \leq n-1$, it is obtained that $a_{(1,m)} c_{(2,1)} \neq c_{(2,m)}$ for all $2 \leq m \leq n-1$. Therefore, we can make a change of coordinates such that

$$f_0(z) = [1 : z : \sum_{i=1}^{\infty} d_{(i,1)} z^{ik} : \sum_{i=2}^{\infty} d_{(i,2)} z^{ik} : \cdots : \sum_{i=2}^{\infty} d_{(i,n-1)} z^{ik}]$$

where $d_{(i,1)} := \frac{c_{(i,1)}}{c_{(1,1)}}$ and $d_{(i,m)} := \frac{c_{(i,m)} - a_{(1,m)} c_{(i,1)}}{c_{(2,m)} - a_{(1,m)} c_{(2,1)}}$, $2 \leq m \leq n-1$. Which proves the case $j = 1$.

We can suppose that the hypothesis is valid for $1 \leq j_0 < n-2$ and that $l_{j_0} = l_{j_0+1} = \cdots = l_{j_0+j_1}$, where $1 \leq j_1 \leq n-2-j_0$ is the largest integer which satisfies this property.

Set $j_2 = j_0 + 2$, $j_3 = j_2 + 1$, $j_4 := j_0 + j_1$ and $j_5 := j_4 + 1$. Using the change of coordinates

$$\varphi([x_0 : \cdots : x_n]) := \begin{cases} [x_0 : \cdots : x_{j_2} : x_{j_3} - x_{j_2} : \cdots : x_{j_4} - x_{j_2} : x_{j_5} : \cdots : x_n] & \text{if } j_4 < n \\ [x_0 : \cdots : x_{j_2} : x_{j_3} - x_{j_2} : \cdots : x_n - x_{j_2}] & \text{if } j_4 = n \end{cases},$$

followed by a permutation in the coordinates of \mathbb{P}^n , it is demonstrated that the hypothesis is valid for $j_0 + 1$. \square

The next theorem describes the hyperosculating points of S and the ramification indexes.

Theorem 9. *Keep the above notations. Then the following is holds:*

- (1) The set of hyperosculating points of S is the set F .
 (2) If $p \in F$, then $b_1(p) = k - 2$ y $b_l(p) = k - 1$ for all $2 \leq l \leq n - 1$

The following corollary is directly derived from Theorem 9.

Corollary 10. *Let z be a local chart of S around the point $p \in S$. Then the normal form of f_0 in $z(p) := 0$ is:*

- (1) If $p \in F$

$$f_0(z) = [1 : z : g_0(z^k) : g_1(z^k) : \cdots : g_i(z^k) : \cdots : g_{n-1}(z^k)]$$

where the g_i are holomorphic functions such that $g_i(z) = z^{i+1} + \cdots + \cdots$,

- (2) If $p \notin F$

$$f_0(z) = [1 : z : z^2 + \cdots : \cdots : z^{(n-1)} + \cdots].$$

Proof of the Theorem 9. Let p be a point in F . Using the Proposition 6 and the Lemma 8, we obtain the following system of equations:

$$\begin{cases} 2 + b_1(p) & = & k \\ 3 + b_1(p) + b_2(p) & = & 2k \\ 4 + b_1(p) + b_2(p) + b_3(p) & = & l_2 k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n + b_1(p) + b_2(p) + b_3(p) + \cdots + b_{n-1}(p) & = & l_{n-2} k \end{cases}$$

Equivalently, we obtain

$$\begin{cases} b_1(p) & = & k - 2 \\ b_2(p) & = & k - 1 \\ b_3(p) & = & (l_2 - 2)k - 1 \\ \vdots & \vdots & \vdots \\ b_{n-1}(p) & = & (l_{n-2} - l_{n-3})k - 1 \end{cases}$$

Observe that $b_l(p) \geq k - 1$ for all $2 \leq l \leq n - 1$. In particular, p is a hyperosculating point.

Since the cardinality of F is equal to $(n + 1)k^{n-1}$, we have the following lower bound form the total ramification indexes:

$$\begin{cases} b_1 & \geq & \hat{b}_1 := (n + 1)k^{n-1}(k - 2) \\ b_l & \geq & \hat{b}_l := (n + 1)k^{n-1}(k - 1) \quad \text{para todo } 2 \leq l \leq n - 1 \end{cases}$$

Observe that in order to finish the demonstration of the theorem, it is necessary and sufficient to prove $b_l = \hat{b}_l$, for all $1 \leq l \leq n - 1$. In that which follows we will prove these equalities.

Consider the following inequality

$$0 \leq b_l - \hat{b}_l \leq \sum_{l=0}^{n-1} (n - l)(b_l - \hat{b}_l),$$

where $b_0 = \hat{b}_0 = 0$. The idea is to show that the right part of the inequality is zero.

Remember that the genus of S is given by the following formula:

$$g := \frac{k^{n-1}((n-1)(k-1)-2)+2}{2}.$$

Via direct calculation, we obtain the following equality:

$$\sum_{l=0}^{n-1} (n - l)\hat{b}_l = n(n + 1)(g - 1) + (n + 1)k^{n-1}.$$

Using de Plucker formulas 5, we obtain

$$\sum_{l=0}^{n-1} (n - l)b_l = \sum_{l=0}^{n-1} (n - l)(2(g - 1) - \Delta^2 d_l) = n(n + 1)(g - 1) - \sum_{l=0}^{n-1} ((n - l)\Delta^2 d_l),$$

where $\Delta^2 d_l = d_{l+1} - 2d_l + d_{l-1}$.

For a simple calculation it is obtained that

$$\sum_{l=0}^{n-1} (n-l) \Delta^2 d_l = d_n - (n+1)d_1 + nd_{-1}.$$

Since $d_n = d_{-1} = 0$ and $d_1 = k^{n-1}$, therefore

$$\sum_{l=0}^{n-1} (n-l) b_l = n(n+1)(g-1) + (n+1)k^{n-1}.$$

Which implies that $b_l = \hat{b}_l$ for all $1 \leq l \leq n-1$. \square

2.3. The weight of the hyperosculating points. Let us conserve the notations from the previous sections. The weight of $p \in S$ is given by the following formula:

$$w(p) := \sum_{i=1}^g (a_i - i)$$

where the a_i are the Weierstrass gaps of p , see Remark 7.

In general, calculating the value of $w(p)$, when p is a Weierstrass point (which is to say $w(p) > 0$), is not an easy problem. In the previous section we proved that the F set is the set of hyperosculating points of S . In this section, we will determine a lower bound for the weight of the hyperosculating points of S .

First let us fix some notations.

Let $I(S) := \langle x_0^k + x_1^k + x_2^k, \dots, \lambda_{n-1} x_0^k + x_1^k + x_{n-1}^k \rangle$ be the homogeneous prime ideal of S in $\mathbb{C}[x_0, \dots, x_n]$, and let $\Gamma(S) := \mathbb{C}[x_0, \dots, x_n]/I(S)$ be the homogeneous coordinate ring of S .

Let $\mathcal{O}_{\mathbb{P}^n}(m)$, $m \in \mathbb{Z}$, be the twisting sheaf. Remember that for $m \geq 0$ the sheaf $\mathcal{O}_{\mathbb{P}^n}(m)$ is generated by the form of degree m of $\mathbb{C}[x_0, \dots, x_n]$. Let us consider the following sheaf over S :

$$\mathcal{O}_S(m) := f_0^* \mathcal{O}_{\mathbb{P}^n}(m),$$

where $f_0 : S \rightarrow \mathbb{P}^n$ is the natural embedding of the generalized Fermat curves. Observe that $H^0(S, \mathcal{O}_S(m)) = \Gamma(S)_m$, where $\Gamma(S)_m$ are the form of degree m of $\Gamma(S)$.

Let $\mathbb{P}(\Gamma(S)_m)$ be the projective space associated to the vectorial space $\Gamma(S)_m$. Observe that the natural linear application $\mathbb{C}[x_0, \dots, x_n]_m \rightarrow \Gamma(S)_m$ induces an embedding. $\mathbb{P}(\Gamma(S)_m) \subset \mathbb{P}(\mathbb{C}[x_0, \dots, x_n]_m) \cong \mathbb{P}^{d(m)}$, $d(m) = \binom{n+m}{m} - 1$.

Let ω_S be the canonical sheaf of S , then $\omega_S \cong \mathcal{O}_S((n-1)(k-1)-2)$ (see page 188 of [Har77]). Then the Veronese application of degree $r := (n-1)(k-1)-2$,

$$\nu_r : \mathbb{P}^n \rightarrow \mathbb{P}^{d(r)},$$

permits us to obtain the canonical embedding f .

$$\begin{array}{ccccc} S & \xrightarrow{f := \nu_r \circ f_0} & \mathbb{P}^n & \xrightarrow{\quad} & \mathbb{P}(\Gamma(S)_r) \hookrightarrow \mathbb{P}^{d(r)} \\ & \searrow f_0 & \swarrow & \searrow \nu_r & \\ & & \mathbb{P}(\Gamma(S)_r) & & \end{array}$$

Using the normal form of f_0 (Corollary 10), we will obtain information about the normal form of canonical embedding f .

Observe that all elements of $p(x_0, x_1, \dots, x_n) \in \Gamma(S)_m$ can be written uniquely in the following form:

$$p(x_0, x_1, \dots, x_n) = \sum_{j=1}^{k-1} x_1^j q_j(x_0, x_2, \dots, x_n)$$

where $q_j(x_0, x_2, \dots, x_n) \in \Gamma(S)_{m-j}$. Which shows that the vectorial space $\Gamma(S)_m$ has the following decomposition:

$$\Gamma(S)_m := \bigoplus_{j=0}^{k-1} x_1^j Q(m-j)$$

where $Q(m-j) \subset \Gamma(S)_{m-j}$.

Proposition 11. *Let us conserve the previously defined notations. Let $s(r-j) := \dim_{\mathbb{C}} Q(r-j)$, where $r := (n-1)(k-1) - 2$ and $0 \leq j \leq k-1$. Then*

$$s(r-j) = \frac{1}{2}k^{n-2}((n(k-1) - 2 - 2j) + \delta_{k-1,j}),$$

where $\delta_{k-1,j}$ is the Kronecker delta.

Proof. Observe that there exists a generalized Fermat curve S' of the type $(n-1, k)$ such that

$$H^0(S', \mathcal{O}_{S'}(r-j)) \cong Q(r-j).$$

For $m \in \mathbb{Z}$, let $h'(m)$ denote the dimension of the global section space $H^0(S', \mathcal{O}_{S'}(m))$ over \mathbb{C} .

Remember that $\omega_{S'} \cong \mathcal{O}_S((n-2)(k-1) - 2)$. Then, using the Riemann-Roch Formula, we obtain

$$h'(-k+1+j) - h(r' - (-k+1+j)) = (-k+1+j)k^{n-2} - \left(\frac{k^{n-2}(((n-2)(k-1)-2)+2)}{2}\right) + 1$$

where $r' := (n-2)(k-1) - 2$.

Since $s(r-j) = h(r' - (-k+1+j))$ y $h'(-k+1+j) = \delta_{k-1,j}$, therefore

$$s(r-j) = \frac{1}{2}k^{n-2}((n(k-1) - 2 - 2j) + \delta_{k-1,j}).$$

□

Now we will estimate the weight of the points $p \in F$.

Theorem 12. *Let p be a point in F , and let $w(p)$ be the weight of the point p . Let us suppose that $n \geq 3$. Then*

$$\widehat{w}(p) := \frac{1}{24}(k-1)(k^{n-1} - 2)(k^n + k^{n-1} - 12) \leq w(p)$$

Remark 13. *If S is a generalized Fermat Curve of the type $(k, 2)$, $k \geq 4$, (which is to say a classic Fermat curve), It is known that (see [Has50], [Roh82] or [Wat99]) the weight of a point $p \in F$ is*

$$w(p) = \frac{1}{24}(k-1)(k-2)(k-3)(k+4).$$

Which shows that the Theorem 12 is valid for the classic case.

Remark 14. *If the vector $(\lambda_1, \dots, \lambda_{n-2})$ is chosen “in a general enough way”, we can expect that $\widehat{w}(p) = w(p)$. For particular choices of the vector $(\lambda_1, \dots, \lambda_{n-2})$, (and n is large enough), surely there exists examples where $\widehat{w}(p) < w(p)$. However have neither proof nor counter example to assure or refute the truth of these affirmations.*

Demonstration of Theorem 12. Let us suppose that $p \in F$. For the Corollary 10, it is obtained that f_0 has the following normal form around of p .

$$f_0(z) = [1 : z : g_1(z^k) : g_2(z^k) : \dots : g_{n-1}(z^k)],$$

where $g_i(z) = z^i + \dots$ for all $0 \leq i \leq n-1$.

Let us suppose that $(n-1)(k-1) - 2 < k$. That which implies that $(n, k) = (3, 3)$ or $(n, k) = (4, 2)$.

If $(n, k) = (3, 3)$, then $H^0(S, \omega_S) \cong H^0(S, \mathcal{O}_S(2)) = \mathbb{C}[x_0, x_1, x_2, x_3]_2$ (forms of degree 2 of the polynomial ring $\mathbb{C}[x_0, x_1, x_2, x_3]$). Then the canonical application f is given by the following composition:

$$S \xrightarrow[f_0]{} \mathbb{P}^3 \xrightarrow[\nu_2]{f} \mathbb{P}^9,$$

Let a_i , $1 \leq i \leq 10$ be the gaps value of $p \in F$. Using the normal form of f_0 in p and the Veronese application ν_2 , we obtain

$$a_i = i, 1 \leq i \leq 5, a_6 = 7, a_7 = 8, a_8 = 10, a_9 = 13 \text{ y } a_{10} \geq 16.$$

In this way we obtain

$$14 \leq w(p).$$

Which finishes the demonstration in this case.

If $(n, k) = (4, 2)$, then $H^0(S, \omega_S) \cong H^0(S, \mathcal{O}_S(1)) = \mathbb{C}[x_0, x_1, x_2, x_3, x_4]_1$. Which shows that the application $S \xrightarrow{f_0} \mathbb{P}^4$ is the canonical application. Using the normal form f_0 in p , we obtain that the gaps values of p are

$$a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 7.$$

In this way we obtain

$$3 \leq w(p).$$

Which finishes the demonstration in this case.

In the rest of the demonstration we assume that $(n-1)(k-1)-2 \geq k$.

Let us consider the normal form of f_0 in p . Observe that there exists a neighborhood $\Omega \subset \mathbb{C}$ of $v = 0$ where the application

$$g : \Omega \rightarrow \mathbb{P}^{n-1} : v \mapsto [1 : g_1(v) : g_2(v) : \cdots : g_{n-1}(v)].$$

is well defined. Let us consider the following application:

$$\Omega \xrightarrow{\nu_{r-j} \circ g} \mathbb{P}^{n-1} \xrightarrow{\quad} \mathbb{P}(Q_{r-j}),$$

where $r = (n-1)(k-1)-2$, $0 \leq j \leq k-1$. Note that this application is well defined.

Considering an automorphism of $\mathbb{P}(Q_{r-j})$ we can obtain the normal form of $\hat{g}_j = \nu_{r-j} \circ g$. That is to say

$$\hat{g}_j(z) = [1 : g_{(1\ j)}(z) : g_{(2\ j)}(z) : \cdots : g_{(t_j\ j)}(z)], \quad t_j := s(r-j) - 1,$$

where the following inequalities are satisfied for each $0 \leq j \leq k-1$:

- $i \leq l_i := \text{Ord}g_{(i\ j)}(v)$, for all $1 \leq i \leq t_j$
- $\text{Ord}g_{(i\ j)}(v) < \text{Ord}g_{(i+1\ j)}(v)$, for all $i \geq 1$

We define the following functions:

$$h_{i\ j}(z) := z^j g_{(i\ j)}(z^k), \quad 0 \leq j \leq k-1, \quad 1 \leq i \leq t_j.$$

Considering an automorphism of \mathbb{P}^{g-1} , where g is the genus of S , we can suppose that the canonical embedding f around the point p is

$$f(z) := [1 : \cdots : z^{k-1} : h_{1\ 0}(z) : \cdots : h_{1\ k-1}(z) : h_{2\ 0}(z) : \cdots : h_{r\ k-2}(z) : h_{r+1\ 0}(z) : \cdots : h_{t_0\ 0}(z)]$$

where $r = t_{k-1} + 1$.

Let a_i , $1 \leq i \leq$ be the gap values of p . Since $ki + j \leq kl_i + j = \text{Ord}h_{i\ j}(z)$, $1 \leq j \leq k-1$ and $1 \leq i \leq t_j$, the following inequality is obtained:

$$\sum_{j=0}^{k-1} \sum_{i=0}^{t_j} (ki + j + 1) \leq \sum_{i=1}^g a_i.$$

Using the Proposition 11, we obtain

$$\sum_{j=0}^{k-1} \sum_{i=0}^{t_j} (ki + j + 1) - \frac{g(g+1)}{2} = \frac{1}{24}(k-1)(k^{n-1}-2)(k^n + k^{n-1} - 12)$$

Which finishes the demonstration, because the weight of the point p is

$$w(p) := \sum_{i=1}^g a_i - \frac{g(g+1)}{2}.$$

□

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